



The Open
University

MST124

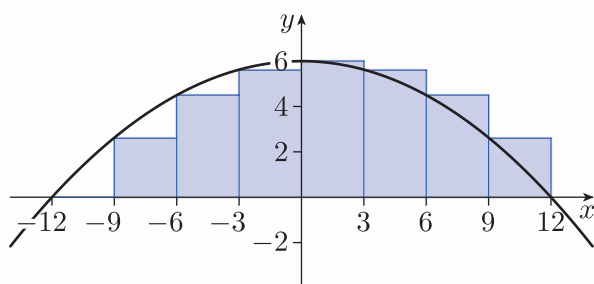
Essential mathematics 1

Exercise Booklet 8

1 Areas, signed areas, and definite integrals

Exercise 1

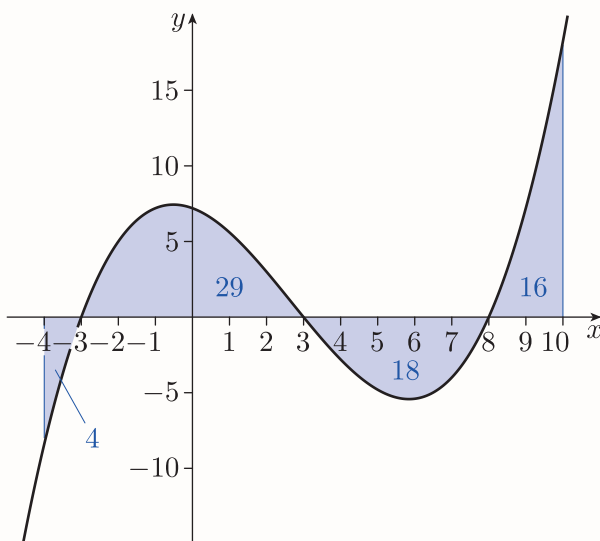
By using eight subintervals as shown below, find an approximate value for the area between the graph of the function $f(x) = 6 - \frac{1}{24}x^2$ and the x -axis.



Exercise 2

The areas of some regions on a graph are marked below.

In each of parts (a)–(h), use these areas to find the signed area between the graph and the x -axis, from the first value of x to the second value of x .

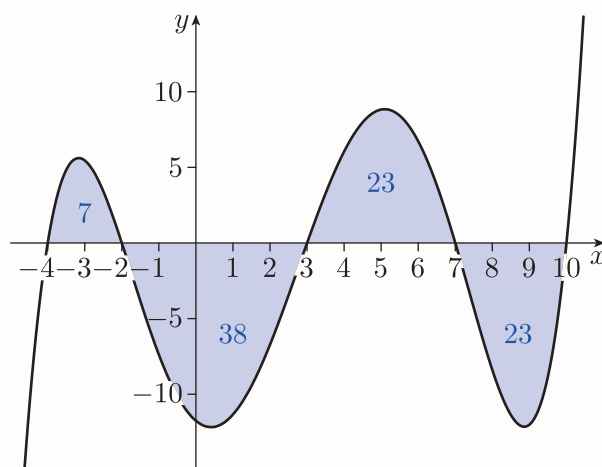


- (a) From $x = -3$ to $x = 3$.
- (b) From $x = -3$ to $x = 8$.
- (c) From $x = 3$ to $x = 10$.
- (d) From $x = -4$ to $x = 3$.
- (e) From $x = -3$ to $x = -4$.
- (f) From $x = 3$ to $x = -4$.
- (g) From $x = 10$ to $x = -3$.
- (h) From $x = -4$ to $x = 10$.

Exercise 3

Consider the function f whose graph is shown below. The areas of some regions are marked. By using these areas, find the values of the following definite integrals.

- (a) $\int_3^7 f(x) \, dx$
- (b) $\int_{-2}^7 f(x) \, dx$
- (c) $\int_{-4}^3 f(x) \, dx$
- (d) $\int_3^{10} f(x) \, dx$
- (e) $\int_{10}^{-2} f(x) \, dx$
- (f) $\int_7^{-4} f(x) \, dx$



2 The fundamental theorem of calculus

Exercise 4

Evaluate each of the following definite integrals. Give exact answers, and also give answers to three significant figures.

- (a) $\int_0^3 5\sqrt{x} \, dx$ (b) $\int_{-1}^1 (x^4 - x^2) \, dx$
- (c) $\int_1^2 \left(1 + \frac{t^4}{24}\right) dt$
- (d) $\int_1^4 \left(\sqrt[3]{x} + 3\sqrt{x} + \frac{1}{x}\right) dx$
- (e) $\int_{\pi/6}^{\pi/3} \operatorname{cosec}^2 \theta \, d\theta$ (f) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2}{1+r^2} dr$

Exercise 5

Evaluate each of the following definite integrals. Give exact answers, and also give answers to three significant figures.

- (a) $\int_1^2 (2x+3)(x-4) \, dx$
- (b) $\int_{-5}^5 (2x+1)^2 \, dx$
- (c) $\int_0^1 \left(\frac{t+1}{\sqrt{t}}\right) dt$
- (d) $\int_0^4 r^2 \left(1 + \frac{1}{r}\right) dr$

Exercise 6

Evaluate each of the following definite integrals. Give exact answers, and also give answers to three significant figures.

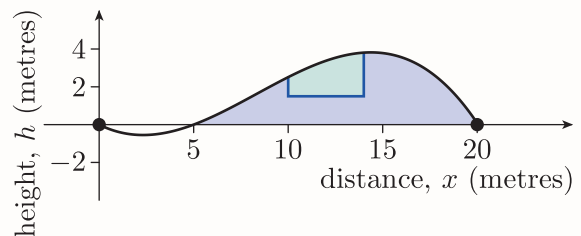
- (a) $\int_0^{\pi/3} (2\cos \theta - \sin \theta) \, d\theta$
- (b) $\int_{-2\pi}^{2\pi} (3\sin x + 2\cos x + 2) \, dx$
- (c) $\int_1^6 \left(e^x + \frac{3}{x}\right) dx$
- (d) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-u^2}} du$

Exercise 7

A cross-section of a particular hill can be represented by the function

$$h = -\frac{x^3}{200} + \frac{x^2}{8} - \frac{x}{2} \quad (0 \leq x \leq 20),$$

where x is distance in metres and h is height above sea level in metres, as shown in the graph below. A team of archaeologists wants to make an excavation to a depth of 1.5 m above sea level between the points corresponding to $x = 10$ and $x = 14$. The excavation is planned to be 2.5 metres across. Assuming that the cross-section of the hill remains the same across these 2.5 metres, find the volume of soil that must be removed, in cubic metres to three significant figures.



Exercise 8

The shape of the logo of a particular company is the shape enclosed by the three curves a , b and c with equations

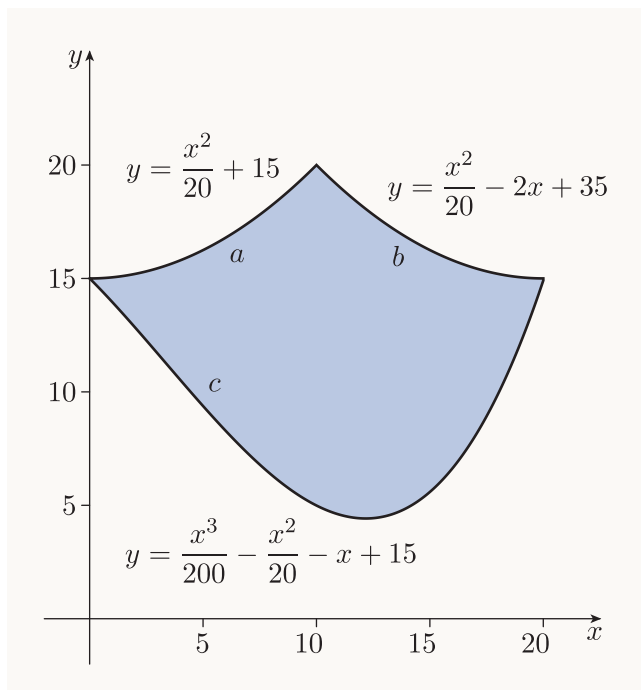
$$y = \frac{x^2}{20} + 15$$

$$y = \frac{x^2}{20} - 2x + 35$$

$$y = \frac{x^3}{200} - \frac{x^2}{20} - x + 15,$$

respectively, as shown on the graph below. The points of intersection of the curves are $(0, 15)$, $(10, 20)$ and $(20, 15)$.

The company plans to commission a large metal display logo for its headquarters. The logo is to be sized as in the graph below, with the axis scales in decimetres (tenths of a metre).

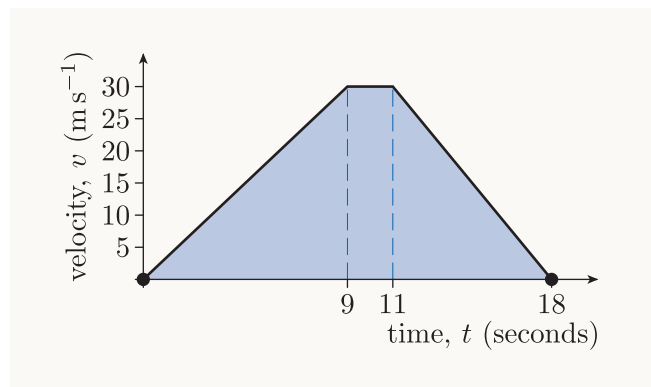


Calculate the area of sheet metal required, as follows.

- Calculate the area between curve *a* and the *x*-axis, from the *y*-axis to its intersection with curve *b*.
- Calculate the area between curve *b* and the *x*-axis, from its intersection with curve *a* to its intersection with curve *c*.
- Calculate the area between curve *c* and the *x*-axis, from the *y*-axis to its intersection with curve *b*.
- Use your answers to parts (a), (b) and (c) to calculate the area of the metal logo, in square decimetres to three significant figures.

Exercise 9

A car moves along a straight road with constant acceleration from rest to a velocity of 30 m s^{-1} , which takes 9 seconds. It then drives at a constant velocity for two seconds, before decelerating at a constant rate to rest again, which takes 7 seconds. The velocity–time graph for its journey is shown below. How far does the car travel in total?

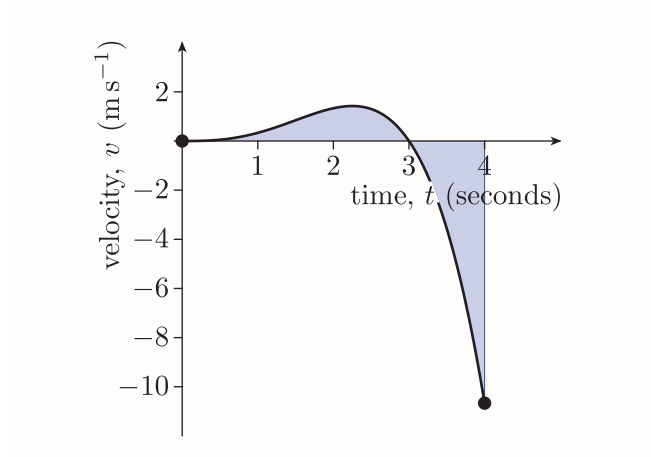


Exercise 10

Suppose that the velocity of an object moving along a straight line is given by the formula

$$v = \frac{1}{6}(3t^3 - t^4) \quad (0 \leq t \leq 4),$$

where *t* is the time in seconds since the object began moving, and *v* is the velocity of the object, in m s^{-1} . The graph of this equation is shown below. It lies above the *t*-axis for $0 < t < 3$, and below it for $3 < t \leq 4$.



Find each of the following, in metres to two decimal places.

- The change in the displacement of the object from time $t = 0$ to time $t = 3$.
- The change in the displacement of the object from time $t = 3$ to time $t = 4$.
- The change in the displacement of the object from time $t = 0$ to time $t = 4$.
- The distance travelled by the object from time $t = 0$ to time $t = 3$.
- The distance travelled by the object from time $t = 3$ to time $t = 4$.
- The total distance travelled by the object from time $t = 0$ to time $t = 4$.

3 Integration by substitution

Exercise 11

Use integration by substitution to find the following integrals.

- (a) $\int x^4 e^{x^5} dx$ (b) $\int x^2 \sin(x^3) dx$
 (c) $\int \frac{x^5}{2-x^6} dx$ (d) $\int x^2(1-2x^3)^9 dx$
 (e) $\int e^x \cos(e^x) dx$ (f) $\int \tan^2 x \sec^2 x dx$

Exercise 12

Find the following integrals.

- (a) $\int (3x^2 + \cos(4x)) dx$
 (b) $\int (e^{2t} - \sin(5t)) dt$
 (c) $\int \left(\frac{e^{3x} + e^{2x}}{e^x} \right) dx$ (d) $\int \frac{4}{1+5x^2} dx$

Exercise 13

Evaluate the following definite integrals. Give exact answers, and also give answers to four decimal places.

- (a) $\int_1^2 (3x^4 - e^{2x}) dx$ (b) $\int_{-\pi}^{\pi} \cos\left(\frac{1}{5}t\right) dt$
 (c) $\int_0^1 e^{t/2} dt$

Exercise 14

Use integration by substitution to find the following integrals.

- (a) $\int (2x-5)^{11} dx$ (b) $\int \sqrt{1-4x} dx$
 (c) $\int \frac{3}{7x-2} dx$

Exercise 15

Use integration by substitution to find the integral

$$\int \cos^7(2x) \sin(2x) dx.$$

Exercise 16

An object is attached to a spring, which hangs vertically from a hook on the ceiling. The object is pulled down to a distance of 0.5 m below the ceiling. It is then released, with initial velocity zero, and oscillates vertically. Assume that the acceleration a (in m s^{-2}) of the object after it starts to oscillate is given by the function

$$a = 2.5 \cos(5t),$$

where t is the time in seconds since the object was released. Take the positive direction along the line of motion to be upwards, and take displacement to be measured from the ceiling (so the displacement of the object at the moment of release is -0.5 m).

- Find an expression for the velocity v (in m s^{-1}) of the object at time t .
- Find an expression for the displacement s (in metres) of the object at time t .
- What is the displacement of the object when it is at its highest position, and at what time does this first occur?
- What is the maximum speed attained by the object, and what is the displacement of the object when this first occurs?

Exercise 17

Use integration by substitution to evaluate the following definite integrals. Give exact answers, and also give answers to four decimal places.

- (a) $\int_0^1 x(2x^2+1)^{1/2} dx$ (b) $\int_0^1 x e^{x^2+3} dx$
 (c) $\int_0^{\pi/2} \frac{\cos x \sin x}{1+\sin^2 x} dx$

Exercise 18

Use integration by substitution to find the following integrals. (In each case, to do this you have to express a further factor in terms of the substitute variable u .)

(a) $\int x\sqrt{x-1} \, dx$

(b) $\int \frac{x}{(3x-1)^4} \, dx$

4 Integration by parts

Exercise 19

Use integration by parts to find each of the following integrals.

(a) $\int x \cos(3x) \, dx$ (b) $\int 2x \sin\left(\frac{1}{5}x\right) \, dx$

(c) $\int x^8 \ln x \, dx$ (d) $\int x \ln(5x) \, dx$

(e) $\int (1-3x) \sin(2x) \, dx$

(f) $\int x^2 \sin\left(\frac{1}{2}x\right) \, dx$

Exercise 20

Find the following integrals.

(a) $\int e^{x/3} \cos(4x) \, dx$

(b) $\int e^{-2x} \sin(5x) \, dx$

Exercise 21

Use integration by parts to evaluate the following definite integrals. Give exact answers, and also answers to three decimal places.

(a) $\int_0^1 x e^{4x} \, dx$ (b) $\int_{-\pi/4}^{\pi/4} x \sin(2x) \, dx$

(c) $\int_1^4 x^2 \ln x \, dx$ (d) $\int_1^2 x^2 \ln(4x) \, dx$

(e) $\int_0^1 x^2 e^{5x} \, dx$

Exercise 22

Evaluate the definite integral

$$\int_0^{\pi/4} e^{x/2} \cos(2x) \, dx.$$

Give an exact answer, and also an answer to four decimal places.

5 More integration

Exercise 23

Find the following integrals, using any appropriate method.

(a) $\int x^2 \sin(4x^3) \, dx$ (b) $\int x \sin(6x) \, dx$

(c) $\int \frac{x^3}{(8-x^4)^6} \, dx$ (d) $\int \frac{5x^3-1}{2x} \, dx$

(e) $\int \cos(4x-5) \, dx$

Exercise 24

Evaluate the following definite integrals, using any appropriate method. Give exact answers, and also answers to four decimal places.

(a) $\int_0^{\pi/6} x \cos(6x) \, dx$ (b) $\int_1^2 \frac{x+2}{(x+1)^2} \, dx$

Solutions to exercises

Solution to Exercise 1

We divide the interval between -12 and 12 into eight subintervals. Each subinterval has width $24/8 = 3$.

The left endpoints of the subintervals are $-12, -9, -6, -3, 0, 3, 6$, and 9 (as shown on the diagram).

So the heights of the rectangles are

$$\begin{aligned} f(-12) &= 6 - \frac{1}{24} \times (-12)^2 = 0 \\ f(-9) &= 6 - \frac{1}{24} \times (-9)^2 = 2.625 \\ f(-6) &= 6 - \frac{1}{24} \times (-6)^2 = 4.5 \\ f(-3) &= 6 - \frac{1}{24} \times (-3)^2 = 5.625 \\ f(0) &= 6 - \frac{1}{24} \times 0^2 = 6 \\ f(3) &= 6 - \frac{1}{24} \times 3^2 = 5.625 \\ f(6) &= 6 - \frac{1}{24} \times 6^2 = 4.5 \\ f(9) &= 6 - \frac{1}{24} \times 9^2 = 2.625. \end{aligned}$$

Hence the approximate area is

$$\begin{aligned} 3 \times (0 + 2.625 + 4.5 + 5.625 \\ + 6 + 5.625 + 4.5 + 2.625) = 94.5. \end{aligned}$$

Solution to Exercise 2

The required signed areas are as follows.

- (a) 29
- (b) $29 - 18 = 11$
- (c) $-18 + 16 = -2$
- (d) $-4 + 29 = 25$
- (e) $-(-4) = 4$
- (f) $-(-4 + 29) = -25$
- (g) $-(29 - 18 + 16) = -27$
- (h) $-4 + 29 - 18 + 16 = 23$

Solution to Exercise 3

- (a) $\int_3^7 f(x) \, dx = 23$
- (b) $\int_{-2}^7 f(x) \, dx = -38 + 23 = -15$
- (c) $\int_{-4}^3 f(x) \, dx = 7 - 38 = -31$
- (d) $\int_3^{10} f(x) \, dx = 23 - 23 = 0$
- (e) $\int_{10}^{-2} f(x) \, dx = -(-38 + 23 - 23) = 38$
- (f) $\int_7^{-4} f(x) \, dx = -(7 - 38 + 23) = 8$

Solution to Exercise 4

- (a)
$$\begin{aligned} \int_0^3 5\sqrt{x} \, dx &= 5 \int_0^3 x^{1/2} \, dx \\ &= 5 \left[\frac{2x^{3/2}}{3} \right]_0^3 \\ &= 5 \left(\frac{2 \times 3^{3/2}}{3} - 0 \right) \\ &= 10\sqrt{3} = 17.3 \text{ (to 3 s.f.)} \end{aligned}$$

- (b)
$$\begin{aligned} \int_{-1}^1 (x^4 - x^2) \, dx &= \left[\frac{x^5}{5} - \frac{x^3}{3} \right]_{-1}^1 \\ &= \left(\frac{1^5}{5} - \frac{1^3}{3} \right) - \left(\frac{(-1)^5}{5} - \frac{(-1)^3}{3} \right) \\ &= \left(\frac{1}{5} - \frac{1}{3} \right) - \left(-\frac{1}{5} + \frac{1}{3} \right) \\ &= \frac{2}{5} - \frac{2}{3} \\ &= -\frac{4}{15} = -0.267 \text{ (to 3 s.f.)} \end{aligned}$$

- (c)
$$\begin{aligned} \int_1^2 \left(1 + \frac{t^4}{24} \right) dt &= \left[t + \frac{t^5}{120} \right]_1^2 \\ &= \left(2 + \frac{2^5}{120} \right) - \left(1 + \frac{1^5}{120} \right) \\ &= 1 + \frac{2^5 - 1}{120} \\ &= \frac{151}{120} = 1.26 \text{ (to 3 s.f.)} \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int_1^4 \left(\sqrt[3]{x} + 3\sqrt{x} + \frac{1}{x} \right) dx \\
 &= \int_1^4 \left(x^{1/3} + 3x^{1/2} + \frac{1}{x} \right) dx \\
 &= \left[\frac{3x^{4/3}}{4} + 2x^{3/2} + \ln|x| \right]_1^4 \\
 &= \left(\frac{3 \times 4^{4/3}}{4} + 2 \times 4^{3/2} + \ln 4 \right) \\
 &\quad - \left(\frac{3 \times 1^{4/3}}{4} + 2 \times 1^{3/2} + \ln 1 \right) \\
 &= \left(3 \times 4^{1/3} + 16 + \ln 4 \right) - \left(\frac{3}{4} + 2 + 0 \right) \\
 &= 3 \times 4^{1/3} + \frac{53}{4} + \ln 4 \\
 &= 19.4 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2 \theta \, d\theta = \left[-\cot \theta \right]_{\pi/6}^{\pi/3} \\
 &= \left[-\frac{1}{\tan \theta} \right]_{\pi/6}^{\pi/3} \\
 &= -\frac{1}{\tan(\pi/3)} + \frac{1}{\tan(\pi/6)} \\
 &= -\frac{1}{\sqrt{3}} + \sqrt{3} \\
 &= \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} \\
 &= \frac{2\sqrt{3}}{3} = 1.15 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2}{1+r^2} \, dr = 2 \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+r^2} \, dr \\
 &= 2 \left[\tan^{-1} r \right]_{1/\sqrt{3}}^{\sqrt{3}} \\
 &= 2 \left(\tan^{-1} \sqrt{3} - \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) \right) \\
 &= 2 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) \\
 &= 2 \times \frac{\pi}{6} \\
 &= \frac{\pi}{3} = 1.05 \text{ (to 3 s.f.)}
 \end{aligned}$$

Solution to Exercise 5

$$\begin{aligned}
 \text{(a)} \quad & \int_1^2 (2x+3)(x-4) \, dx \\
 &= \int_1^2 (2x^2 - 5x - 12) \, dx \\
 &= \left[\frac{2}{3}x^3 - \frac{5}{2}x^2 - 12x \right]_1^2 \\
 &= \left(\frac{2}{3} \times 2^3 - \frac{5}{2} \times 2^2 - 12 \times 2 \right) \\
 &\quad - \left(\frac{2}{3} - \frac{5}{2} - 12 \right) \\
 &= -\frac{89}{6} = -14.8 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_{-5}^5 (2x+1)^2 \, dx \\
 &= \int_{-5}^5 (4x^2 + 4x + 1) \, dx \\
 &= \left[\frac{4}{3}x^3 + 2x^2 + x \right]_{-5}^5 \\
 &= \left(\frac{4}{3} \times 5^3 + 2 \times 5^2 + 5 \right) \\
 &\quad - \left(\frac{4}{3}(-5)^3 + 2(-5)^2 + (-5) \right) \\
 &= \frac{1030}{3} = 343 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int_0^1 \left(\frac{t+1}{\sqrt{t}} \right) dt \\
 &= \int_0^1 \left(\frac{t}{t^{1/2}} + \frac{1}{t^{1/2}} \right) dt \\
 &= \int_0^1 \left(t^{1/2} + t^{-1/2} \right) dt \\
 &= \left[\frac{2}{3}t^{3/2} + 2t^{1/2} \right]_0^1 \\
 &= \left(\frac{2}{3} + 2 \right) - 0 \\
 &= \frac{8}{3} = 2.67 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int_0^4 r^2 \left(1 + \frac{1}{r} \right) dr \\
 &= \int_0^4 (r^2 + r) \, dr \\
 &= \left[\frac{1}{3}r^3 + \frac{1}{2}r^2 \right]_0^4 \\
 &= \left(\frac{1}{3} \times 4^3 + \frac{1}{2} \times 4^2 \right) - 0 \\
 &= \frac{88}{3} = 29.3 \text{ (to 3 s.f.)}
 \end{aligned}$$

Solution to Exercise 6

$$\begin{aligned}
 \text{(a)} \quad & \int_0^{\pi/3} (2 \cos \theta - \sin \theta) \, d\theta \\
 &= \left[2 \sin \theta + \cos \theta \right]_0^{\pi/3} \\
 &= \left(2 \sin \left(\frac{\pi}{3} \right) + \cos \left(\frac{\pi}{3} \right) \right) \\
 &\quad - (2 \sin 0 + \cos 0) \\
 &= \left(2 \times \frac{\sqrt{3}}{2} + \frac{1}{2} \right) - (0 + 1) \\
 &= \sqrt{3} - \frac{1}{2} = 1.23 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_{-2\pi}^{2\pi} (3 \sin x + 2 \cos x + 2) \, dx \\
 &= \left[-3 \cos x + 2 \sin x + 2x \right]_{-2\pi}^{2\pi} \\
 &= (-3 \cos(2\pi) + 2 \sin(2\pi) + 2 \times 2\pi) \\
 &\quad - (-3 \cos(-2\pi) + 2 \sin(-2\pi) + 2 \times (-2\pi)) \\
 &= (-3 \times 1 + 2 \times 0 + 4\pi) \\
 &\quad - (-3 \times 1 + 2 \times 0 - 4\pi) \\
 &= 8\pi = 25.1 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int_1^6 \left(e^x + \frac{3}{x} \right) \, dx \\
 &= \left[e^x + 3 \ln x \right]_1^6 \\
 &= (e^6 + 3 \ln 6) - (e^1 + 3 \ln 1) \\
 &= e^6 - e + 3 \ln 6 \\
 &= 406 \text{ (to 3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-u^2}} \, du \\
 &= \left[\sin^{-1} u \right]_{-1/2}^{1/2} \\
 &= \sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} \left(-\frac{1}{2} \right) \\
 &= \pi/6 - (-\pi/6) \\
 &= \pi/3 \\
 &= 1.05 \text{ (to 3 s.f.)}
 \end{aligned}$$

Solution to Exercise 7

We have

$$\begin{aligned}
 & \int_{10}^{14} \left(-\frac{x^3}{200} + \frac{x^2}{8} - \frac{x}{2} \right) \, dx \\
 &= \left[-\frac{x^4}{800} + \frac{x^3}{24} - \frac{x^2}{4} \right]_{10}^{14} \\
 &= \left(-\frac{14^4}{800} + \frac{14^3}{24} - \frac{14^2}{4} \right) \\
 &\quad - \left(-\frac{10^4}{800} + \frac{10^3}{24} - \frac{10^2}{4} \right) \\
 &= \frac{986}{75}.
 \end{aligned}$$

This is the area in square metres between the curve and the x -axis, from $x = 10$ to $x = 14$.

To find the area of the side of the excavation, we have to subtract the part of this area that is not to be removed, which gives

$$\frac{986}{75} - (1.5 \times 4) = \frac{536}{75}.$$

Hence the volume of soil that is to be removed is given by

$$\frac{536}{75} \times 2.5 = \frac{268}{15} = 17.9 \text{ (to 3 s.f.)}.$$

So approximately 17.9 cubic metres of soil must be removed.

Solution to Exercise 8

(a) The required area is

$$\begin{aligned}
 & \int_0^{10} \left(\frac{x^2}{20} + 15 \right) \, dx \\
 &= \left[\frac{x^3}{60} + 15x \right]_0^{10} \\
 &= \left(\frac{10^3}{60} + 15 \times 10 \right) - 0 \\
 &= \frac{500}{3}.
 \end{aligned}$$

(b) The required area is

$$\begin{aligned}
 & \int_{10}^{20} \left(\frac{x^2}{20} - 2x + 35 \right) \, dx \\
 &= \left[\frac{x^3}{60} - x^2 + 35x \right]_{10}^{20} \\
 &= \left(\frac{20^3}{60} - 20^2 + 35 \times 20 \right) \\
 &\quad - \left(\frac{10^3}{60} - 10^2 + 35 \times 10 \right) \\
 &= \frac{500}{3}.
 \end{aligned}$$

Exercise Booklet 8

- (c) The required area is

$$\begin{aligned} & \int_0^{20} \left(\frac{x^3}{200} - \frac{x^2}{20} - x + 15 \right) dx \\ &= \left[\frac{x^4}{800} - \frac{x^3}{60} - \frac{x^2}{2} + 15x \right]_0^{20} \\ &= \left(\frac{20^4}{800} - \frac{20^3}{60} - \frac{20^2}{2} + 15 \times 20 \right) - 0 \\ &= \frac{500}{3}. \end{aligned}$$

- (d) You can see from the graph that you can find the area of the logo by adding together the two areas obtained in parts (a) and (b) and subtracting the area found in part (c). So the area, in square decimetres, is

$$2 \times \frac{500}{3} - \frac{500}{3} = \frac{500}{3} = 167 \text{ (to 3 s.f.)}.$$

Solution to Exercise 9

The distance travelled by the car is the area between the velocity–time graph and the time axis, which is

$$\begin{aligned} & \frac{1}{2} \times 9 \times 30 + 2 \times 30 + \frac{1}{2} \times 7 \times 30 \\ &= 135 + 60 + 105 \\ &= 300 \text{ m.} \end{aligned}$$

Solution to Exercise 10

- (a) The change in the displacement of the object from time $t = 0$ to time $t = 3$ is given by

$$\begin{aligned} & \int_0^3 \frac{1}{6}(3t^3 - t^4) dt \\ &= \frac{1}{6} \int_0^3 (3t^3 - t^4) dt \\ &= \frac{1}{6} \left[\frac{3}{4}t^4 - \frac{1}{5}t^5 \right]_0^3 \\ &= \frac{1}{6} \left(\frac{3}{4} \times 3^4 - \frac{1}{5} \times 3^5 \right) \\ &= \frac{81}{40} = 2.03 \text{ (to 2 d.p.)}. \end{aligned}$$

So the change in displacement is 2.03 m (to 2 d.p.).

- (b) The change in the displacement of the object from time $t = 3$ to time $t = 4$ is given by

$$\begin{aligned} & \int_3^4 \frac{1}{6}(3t^3 - t^4) dt \\ &= \frac{1}{6} \int_3^4 (3t^3 - t^4) dt \\ &= \frac{1}{6} \left[\frac{3}{4}t^4 - \frac{1}{5}t^5 \right]_3^4 \\ &= \frac{1}{6} \left(\left(\frac{3}{4} \times 4^4 - \frac{1}{5} \times 4^5 \right) - \left(\frac{3}{4} \times 3^4 - \frac{1}{5} \times 3^5 \right) \right) \\ &= -\frac{499}{120} = -4.16 \text{ (to 2 d.p.)}. \end{aligned}$$

So the change in displacement is -4.16 m (to 2 d.p.).

- (c) The change in the displacement of the object from time $t = 0$ to time $t = 4$ is given by

$$\int_0^4 \frac{1}{6}(3t^3 - t^4) dt.$$

This integral is equal to

$$\int_0^3 \frac{1}{6}(3t^3 - t^4) dt + \int_3^4 \frac{1}{6}(3t^3 - t^4) dt,$$

which, from parts (a) and (b), is equal to

$$\frac{81}{40} - \frac{499}{120} = -\frac{32}{15} = -2.13 \text{ (to 2 d.p.)}.$$

So the change in displacement is -2.13 m (to 2 d.p.).

- (d) The distance travelled by the object from time $t = 0$ to time $t = 3$ is the magnitude of the answer to part (a), which is 2.03 m (to 2 d.p.).
- (e) The distance travelled by the object from time $t = 3$ to time $t = 4$ is the magnitude of the answer to part (b), which is 4.16 m (to 2 d.p.).
- (f) The total distance travelled by the object from time $t = 0$ to time $t = 4$ is, by the solutions to parts (d) and (e), given by

$$\frac{81}{40} + \frac{499}{120} = \frac{371}{60} = 6.18 \text{ (to 2 d.p.)}.$$

So the total distance travelled is 6.18 m (to 2 d.p.).

Solution to Exercise 11

(a) Let $u = x^5$; then $\frac{du}{dx} = 5x^4$. So

$$\begin{aligned}\int x^4 e^{x^5} dx &= \frac{1}{5} \int e^{x^5} (5x^4) dx \\ &= \frac{1}{5} \int e^u du \\ &= \frac{1}{5} e^u + c \\ &= \frac{1}{5} e^{x^5} + c.\end{aligned}$$

(b) Let $u = x^3$; then $\frac{du}{dx} = 3x^2$. So

$$\begin{aligned}\int x^2 \sin(x^3) dx &= \frac{1}{3} \int \sin(x^3) (3x^2) dx \\ &= \frac{1}{3} \int \sin u du \\ &= -\frac{1}{3} \cos u + c \\ &= -\frac{1}{3} \cos(x^3) + c.\end{aligned}$$

(c) Let $u = 2 - x^6$; then $\frac{du}{dx} = -6x^5$. So

$$\begin{aligned}\int \frac{x^5}{2 - x^6} dx &= -\frac{1}{6} \int \frac{1}{2 - x^6} \times (-6x^5) dx \\ &= -\frac{1}{6} \int \frac{1}{u} du \\ &= -\frac{1}{6} \ln |u| + c \\ &= -\frac{1}{6} \ln |2 - x^6| + c.\end{aligned}$$

(d) Let $u = 1 - 2x^3$; then $\frac{du}{dx} = -6x^2$. So

$$\begin{aligned}\int x^2 (1 - 2x^3)^9 dx &= -\frac{1}{6} \int (1 - 2x^3)^9 (-6x^2) dx \\ &= -\frac{1}{6} \int u^9 du \\ &= -\frac{1}{6} \times \frac{1}{10} u^{10} + c \\ &= -\frac{1}{60} (1 - 2x^3)^{10} + c.\end{aligned}$$

(e) Let $u = e^x$; then $\frac{du}{dx} = e^x$. So

$$\begin{aligned}\int e^x \cos(e^x) dx &= \int (\cos(e^x)) (e^x) dx \\ &= \int \cos u du \\ &= \sin u + c \\ &= \sin(e^x) + c.\end{aligned}$$

(f) Let $u = \tan x$; then $\frac{du}{dx} = \sec^2 x$. So

$$\begin{aligned}\int \tan^2 x \sec^2 x dx &= \int u^2 du \\ &= \frac{1}{3} u^3 + c \\ &= \frac{1}{3} \tan^3 x + c.\end{aligned}$$

Solution to Exercise 12

(a) By the sum rule and the rule for integrating a function of a linear expression,

$$\begin{aligned}\int (3x^2 + \cos(4x)) dx &= x^3 + \frac{1}{4} \sin(4x) + c.\end{aligned}$$

(b) By the sum rule and the rule for integrating a function of a linear expression,

$$\begin{aligned}\int (e^{2t} - \sin(5t)) dt &= \frac{1}{2} e^{2t} + \frac{1}{5} \cos(5t) + c.\end{aligned}$$

(c) Expanding the fraction in the integrand gives

$$\begin{aligned}\int \left(\frac{e^{3x} + e^{2x}}{e^x} \right) dx &= \int \left(\frac{e^{3x}}{e^x} + \frac{e^{2x}}{e^x} \right) dx \\ &= \int (e^{2x} + e^x) dx \\ &= \frac{1}{2} e^{2x} + e^x + c.\end{aligned}$$

(d) Let $u = \sqrt{5}x$; then $du/dx = \sqrt{5}$. So

$$\begin{aligned} & \int \frac{4}{1+5x^2} dx \\ &= \frac{4}{\sqrt{5}} \int \frac{1}{1+(\sqrt{5}x)^2} \sqrt{5} dx \\ &= \frac{4}{\sqrt{5}} \int \frac{1}{1+u^2} du \\ &= \frac{4}{\sqrt{5}} \tan^{-1} u + c \\ &= \frac{4}{\sqrt{5}} \tan^{-1} (\sqrt{5}x) + c. \end{aligned}$$

Solution to Exercise 13

$$\begin{aligned} \text{(a)} \quad & \int_1^2 (3x^4 - e^{2x}) dx \\ &= \left[\frac{3}{5}x^5 - \frac{1}{2}e^{2x} \right]_1^2 \\ &= \left(\frac{3}{5} \times 32 - \frac{1}{2} \times e^4 \right) - \left(\frac{3}{5} - \frac{1}{2}e^2 \right) \\ &= \frac{3}{5} \times 31 + \frac{1}{2}(e^2 - e^4) \\ &= \frac{1}{10}(5e^2 - 5e^4 + 186) \\ &= -5.0045 \text{ (to 4 d.p.)} \\ \text{(b)} \quad & \int_{-\pi}^{\pi} \cos\left(\frac{1}{5}t\right) dt = \left[5 \sin\left(\frac{1}{5}t\right) \right]_{-\pi}^{\pi} \\ &= 5 \left(\sin\left(\frac{\pi}{5}\right) - \sin\left(-\frac{\pi}{5}\right) \right) \\ &= 10 \sin\left(\frac{\pi}{5}\right) \\ &= 5.8779 \text{ (to 4 d.p.)} \\ \text{(c)} \quad & \int_0^1 e^{t/2} dt = \left[2e^{t/2} \right]_0^1 \\ &= 2e^{1/2} - 2 \\ &= 1.2974 \text{ (to 4 d.p.)} \end{aligned}$$

Solution to Exercise 14

(a) Let $u = 2x - 5$; then $du/dx = 2$. So

$$\begin{aligned} & \int (2x - 5)^{11} dx \\ &= \frac{1}{2} \int (2x - 5)^{11} \times 2 dx \\ &= \frac{1}{2} \int u^{11} du \\ &= \frac{1}{2} \times \frac{1}{12} u^{12} + c \\ &= \frac{1}{24} (2x - 5)^{12} + c. \end{aligned}$$

(b) Let $u = 1 - 4x$; then $du/dx = -4$. So

$$\begin{aligned} & \int \sqrt{1-4x} dx \\ &= -\frac{1}{4} \int (1-4x)^{1/2} \times (-4) dx \\ &= -\frac{1}{4} \int u^{1/2} du \\ &= -\frac{1}{4} \times \frac{2}{3} u^{3/2} + c \\ &= -\frac{1}{6} (1-4x)^{3/2} + c. \end{aligned}$$

(c) Let $u = 7x - 2$; then $du/dx = 7$. So

$$\begin{aligned} & \int \frac{3}{7x-2} dx \\ &= \frac{3}{7} \int \frac{1}{7x-2} \times 7 dx \\ &= \frac{3}{7} \int \frac{1}{u} du \\ &= \frac{3}{7} \ln |u| + c \\ &= \frac{3}{7} \ln |7x-2| + c. \end{aligned}$$

(You may be able to find the integrals in this question more quickly by using the rule for integrating a function of a linear expression.)

Solution to Exercise 15

Let $u = \cos(2x)$; then $\frac{du}{dx} = -2 \sin(2x)$. So

$$\begin{aligned} & \int \cos^7(2x) \sin(2x) dx \\ &= -\frac{1}{2} \int \cos^7(2x) (-2 \sin(2x)) dx \\ &= -\frac{1}{2} \int u^7 du \\ &= -\frac{1}{2} \times \frac{1}{8} u^8 + c \\ &= -\frac{1}{16} \cos^8(2x) + c. \end{aligned}$$

Solution to Exercise 16

(a) The acceleration of the object is given by

$$a = 2.5 \cos(5t).$$

Hence its velocity is given by

$$\begin{aligned} v &= \int (2.5 \cos(5t)) dt \\ &= \frac{2.5}{5} \sin(5t) + c \\ &= 0.5 \sin(5t) + c, \end{aligned}$$

where c is a constant.

Since $v = 0$ when $t = 0$, we have

$$0 = 0.5 \sin 0 + c$$

$$0 = 0 + c$$

$$c = 0.$$

Hence the velocity is given by

$$v = 0.5 \sin(5t).$$

- (b) By part (a), the displacement of the object is given by

$$s = \int (0.5 \sin(5t)) dt$$

$$= -\frac{0.5}{5} \cos(5t) + k$$

$$= -0.1 \cos(5t) + k,$$

where k is a constant. Since $s = -0.5$ when $t = 0$, we have

$$-0.5 = -0.1 \cos(5t) + k$$

$$-0.5 = -0.1 \times 1 + k$$

$$k = -0.4.$$

Hence the displacement of the object is given by

$$s = -0.1 \cos(5t) - 0.4.$$

- (c) Since $\cos(5t)$ takes values between -1 and 1 , the displacement (in metres) of the object takes values between

$$-0.1 \times (-1) - 0.4$$

and

$$-0.1 \times 1 - 0.4;$$

that is, between -0.3 and -0.5 .

The object is at its highest position when its displacement is -0.3 m.

This displacement occurs at every value of t such that

$$\cos(5t) = -1.$$

So it occurs for the first time when

$$5t = \pi;$$

that is, when

$$t = \frac{\pi}{5} = 0.628 \text{ (to 3 s.f.)}.$$

So this displacement first occurs 0.628 seconds (to 3 s.f.) after the object is released.

- (d) From part (a), the velocity of the object is given by

$$v = 0.5 \sin(5t).$$

Hence its speed is given by

$$|v| = |0.5 \sin(5t)|.$$

Since $\sin(5t)$ takes values between -1 and 1 , the greatest speed is given by

$$|v| = 0.5 \times 1 = 0.5.$$

So the maximum speed is 0.5 m s^{-1} .

This maximum speed occurs at every value of t such that

$$|\sin(5t)| = 1;$$

that is, such that

$$\sin(5t) = \pm 1.$$

So it occurs for the first time when

$$5t = \frac{\pi}{2};$$

that is,

$$t = \frac{\pi}{10}.$$

This gives

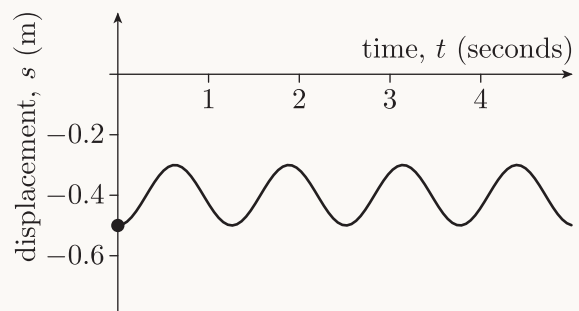
$$s = -0.1 \cos\left(5 \times \frac{\pi}{10}\right) - 0.4$$

$$= -0.1 \cos\left(\frac{\pi}{2}\right) - 0.4$$

$$= -0.4.$$

So the displacement of the object when the maximum speed first occurs is -0.4 m.

(The displacement–time graph for the object is shown below. The highest position seems to first occur after about 0.6 seconds, as expected from the answer found to part (c), and the maximum speed (the largest magnitude of the gradient) seems to first occur when the object's displacement is -0.4 m, as expected from the answer found to part (d). In fact, the maximum speed occurs whenever the object's displacement is -0.4 m.)



Solution to Exercise 17

- (a) Let $u = 2x^2 + 1$; then $\frac{du}{dx} = 4x$.

Also, $x = 0$ gives $u = 1$ and $x = 1$ gives $u = 3$. Hence

$$\begin{aligned} \int_0^1 x(2x^2 + 1)^{1/2} dx &= \frac{1}{4} \int_0^1 (2x^2 + 1)^{1/2} (4x) dx \\ &= \frac{1}{4} \int_1^3 u^{1/2} du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{3/2} \right]_1^3 \\ &= \frac{1}{6} (3^{3/2} - 1^{3/2}) \\ &= \frac{1}{6} (3\sqrt{3} - 1) \\ &= 0.6994 \text{ (to 4 d.p.)}. \end{aligned}$$

- (b) Take $u = x^2 + 3$ so $\frac{du}{dx} = 2x$.

Also, $x = 0$ gives $u = 3$, and $x = 1$ gives $u = 4$. Hence

$$\begin{aligned} \int_0^1 x e^{x^2+3} dx &= \frac{1}{2} \int_0^1 e^{x^2+3} (2x) dx \\ &= \frac{1}{2} \int_3^4 e^u du \\ &= \frac{1}{2} [e^u]_3^4 \\ &= \frac{1}{2} (e^4 - e^3) \\ &= \frac{1}{2} e^3 (e - 1) \\ &= 17.2563 \text{ (to 4 d.p.)}. \end{aligned}$$

- (c) Let $u = 1 + \sin^2 x$; then $\frac{du}{dx} = 2 \sin x \cos x$.

Also, $x = 0$ gives $u = 1$ and $x = \pi/2$ gives $u = 2$. Hence

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos x \sin x}{1 + \sin^2 x} dx &= \frac{1}{2} \int_0^{\pi/2} \frac{1}{1 + \sin^2 x} (2 \cos x \sin x) dx \\ &= \frac{1}{2} \int_1^2 \frac{1}{u} du \\ &= \frac{1}{2} [\ln u]_1^2 \\ &= \frac{1}{2} (\ln 2 - \ln 1) \\ &= \frac{1}{2} \ln 2 \\ &= 0.3466 \text{ (to 4 d.p.)}. \end{aligned}$$

Solution to Exercise 18

- (a) Let $u = x - 1$; then $\frac{du}{dx} = 1$.

Also, rearranging the equation $u = x - 1$ gives $x = u + 1$.

Hence

$$\begin{aligned} \int x \sqrt{x-1} dx &= \int (u+1) u^{1/2} du \\ &= \int (u^{3/2} + u^{1/2}) du \\ &= \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + c \\ &= \frac{2}{5} (x-1)^{5/2} + \frac{2}{3} (x-1)^{3/2} + c. \end{aligned}$$

(b) Let $u = 3x - 1$; then $\frac{du}{dx} = 3$.

Also, rearranging the equation $u = 3x - 1$ gives

$$\begin{aligned} 3x &= u + 1 \\ x &= \frac{1}{3}(u + 1). \end{aligned}$$

Hence

$$\begin{aligned} \int \frac{x}{(3x-1)^4} dx &= \frac{1}{3} \int \frac{x}{(3x-1)^4} \times 3 dx \\ &= \frac{1}{3} \int \frac{\frac{1}{3}(u+1)}{u^4} du \\ &= \frac{1}{9} \int (u^{-3} + u^{-4}) du \\ &= \frac{1}{9} \left(-\frac{1}{2}u^{-2} - \frac{1}{3}u^{-3} \right) + c \\ &= -\frac{1}{9} \left(\frac{1}{2u^2} + \frac{1}{3u^3} \right) + c \\ &= -\frac{1}{9} \times \frac{3u+2}{6u^3} + c \\ &= -\frac{3u+2}{54u^3} + c \\ &= -\frac{3(3x-1)+2}{54(3x-1)^3} + c \\ &= -\frac{(9x-1)}{54(3x-1)^3} + c. \end{aligned}$$

Solution to Exercise 19

(a) $\int x \cos(3x) dx$

$$\begin{aligned} &= x \times \frac{1}{3} \sin(3x) - \int 1 \times \frac{1}{3} \sin(3x) dx \\ &= \frac{1}{3}x \sin(3x) - \frac{1}{3} \int \sin(3x) dx \\ &= \frac{1}{3}x \sin(3x) + \frac{1}{9} \cos(3x) + c \end{aligned}$$

(b) $\int 2x \sin\left(\frac{1}{5}x\right) dx$

$$\begin{aligned} &= 2x \left(-5 \cos\left(\frac{1}{5}x\right) \right) - \int 2 \left(-5 \cos\left(\frac{1}{5}x\right) \right) dx \\ &= -10x \cos\left(\frac{1}{5}x\right) + 10 \int \cos\left(\frac{1}{5}x\right) dx \\ &= -10x \cos\left(\frac{1}{5}x\right) + 50 \sin\left(\frac{1}{5}x\right) + c \end{aligned}$$

(c) $\int x^8 \ln x dx$

$$\begin{aligned} &= \int (\ln x) x^8 dx \\ &= (\ln x) \times \frac{1}{9}x^9 - \int \frac{1}{x} \times \frac{1}{9}x^9 dx \\ &= \frac{1}{9}x^9 \ln x - \frac{1}{9} \int x^8 dx \\ &= \frac{1}{9}x^9 \ln x - \frac{1}{9} \times \frac{1}{9}x^9 + c \\ &= \frac{1}{9}x^9 \ln x - \frac{1}{81}x^9 + c \\ &= \frac{1}{81}x^9(9 \ln x - 1) + c \end{aligned}$$

(d) $\int x \ln(5x) dx$

$$\begin{aligned} &= \int (\ln(5x)) x dx \\ &= \ln(5x) \times \frac{1}{2}x^2 - \int \frac{1}{x} \times \frac{1}{2}x^2 dx \\ &= \frac{1}{2}x^2 \ln(5x) - \frac{1}{2} \int x dx \\ &= \frac{1}{2}x^2 \ln(5x) - \frac{1}{4}x^2 + c \\ &= \frac{1}{4}x^2 (2 \ln(5x) - 1) + c \end{aligned}$$

(e) $\int (1-3x) \sin(2x) dx$

$$\begin{aligned} &= (1-3x) \left(-\frac{1}{2} \cos(2x) \right) \\ &\quad - \int (-3) \left(-\frac{1}{2} \cos(2x) \right) dx \\ &= \frac{1}{2}(3x-1) \cos(2x) - \frac{3}{2} \int \cos(2x) dx \\ &= \frac{1}{2}(3x-1) \cos(2x) - \frac{3}{2} \times \frac{1}{2} \sin(2x) + c \\ &= \frac{1}{2}(3x-1) \cos(2x) - \frac{3}{4} \sin(2x) + c \end{aligned}$$

(f) In this part we integrate by parts twice.

$$\begin{aligned} &\int x^2 \sin\left(\frac{1}{2}x\right) dx \\ &= x^2 \left(-2 \cos\left(\frac{1}{2}x\right) \right) - \int 2x \left(-2 \cos\left(\frac{1}{2}x\right) \right) dx \\ &= -2x^2 \cos\left(\frac{1}{2}x\right) + 4 \int x \cos\left(\frac{1}{2}x\right) dx \\ &= -2x^2 \cos\left(\frac{1}{2}x\right) \\ &\quad + 4 \left(x \times 2 \sin\left(\frac{1}{2}x\right) - \int 1 \times 2 \sin\left(\frac{1}{2}x\right) dx \right) + c \\ &= -2x^2 \cos\left(\frac{1}{2}x\right) \\ &\quad + 8x \sin\left(\frac{1}{2}x\right) - 8 \int \sin\left(\frac{1}{2}x\right) dx + c \\ &= -2x^2 \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) + 16 \cos\left(\frac{1}{2}x\right) + c \\ &= 2(8-x^2) \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) + c \end{aligned}$$

Solution to Exercise 20

In each part we integrate by parts twice.

- (a) We can begin by swapping the order of the two factors in the integrand, to avoid introducing fractions in the early stages:

$$\int e^{x/3} \cos(4x) \, dx = \int \cos(4x) e^{x/3} \, dx.$$

Integrating by parts twice gives

$$\begin{aligned} & \int \cos(4x) e^{x/3} \, dx \\ &= \cos(4x) \times 3e^{x/3} - \int (-4 \sin(4x) \times 3e^{x/3}) \, dx \\ &= 3 \cos(4x) e^{x/3} + 12 \int (\sin(4x) e^{x/3}) \, dx \\ &= 3 \cos(4x) e^{x/3} + 12 \left(\sin(4x) \times 3e^{x/3} \right. \\ & \quad \left. - \int 4 \cos(4x) \times 3e^{x/3} \, dx \right) \\ &= 3 \cos(4x) e^{x/3} + 36 \sin(4x) e^{x/3} \\ & \quad - 144 \int \cos(4x) e^{x/3} \, dx. \end{aligned}$$

Hence

$$\begin{aligned} & 145 \int \cos(4x) e^{x/3} \, dx \\ &= 3 \cos(4x) e^{x/3} + 36 \sin(4x) e^{x/3} + c, \end{aligned}$$

which gives

$$\begin{aligned} & \int \cos(4x) e^{x/3} \, dx \\ &= \frac{1}{145} \left(3 \cos(4x) e^{x/3} + 36 \sin(4x) e^{x/3} \right) + c. \end{aligned}$$

(Remember that there is no need to divide the arbitrary constant c by 145: we just use a different arbitrary constant, which we also denote by c .)

Swapping the order of the factors in the left-hand side back again, and taking out the common factors in the right-hand side, gives

$$\begin{aligned} & \int e^{x/3} \cos(4x) \, dx \\ &= \frac{3}{145} e^{x/3} (12 \sin(4x) + \cos(4x)) + c. \end{aligned}$$

- (b) (In this part, swapping the order of the factors won't lead to fractions being avoided in the early stages, so we leave the order as it is.)

Integrating by parts twice gives

$$\begin{aligned} & \int e^{-2x} \sin(5x) \, dx \\ &= e^{-2x} \left(-\frac{1}{5} \cos(5x) \right) \\ & \quad - \int (-2e^{-2x}) \left(-\frac{1}{5} \cos(5x) \right) \, dx \\ &= -\frac{1}{5} e^{-2x} \cos(5x) - \frac{2}{5} \int e^{-2x} \cos(5x) \, dx \\ &= -\frac{1}{5} e^{-2x} \cos(5x) - \frac{2}{5} \left(e^{-2x} \left(\frac{1}{5} \sin(5x) \right) \right. \\ & \quad \left. - \int (-2e^{-2x}) \left(\frac{1}{5} \sin(5x) \right) \, dx \right) \\ &= -\frac{1}{5} e^{-2x} \cos(5x) - \frac{2}{25} e^{-2x} \sin(5x) \\ & \quad - \frac{4}{25} \int e^{-2x} \sin(5x) \, dx. \end{aligned}$$

Hence

$$\begin{aligned} & \left(1 + \frac{4}{25} \right) \int e^{-2x} \sin(5x) \, dx \\ &= -\frac{1}{5} e^{-2x} \cos(5x) - \frac{2}{25} e^{-2x} \sin(5x) + c, \end{aligned}$$

which gives

$$\begin{aligned} & \int e^{-2x} \sin(5x) \, dx \\ &= \frac{25}{29} \left(-\frac{1}{5} e^{-2x} \cos(5x) - \frac{2}{25} e^{-2x} \sin(5x) \right) + c \\ &= -\frac{1}{29} e^{-2x} (5 \cos(5x) + 2 \sin(5x)) + c. \end{aligned}$$

(Similarly to part (a), there is no need to divide the arbitrary constant c by $\frac{29}{25}$.)

Solution to Exercise 21

$$\begin{aligned} & \text{(a) } \int_0^1 x e^{4x} \, dx \\ &= \left[x \times \frac{1}{4} e^{4x} \right]_0^1 - \int_0^1 1 \times \frac{1}{4} e^{4x} \, dx \\ &= \left[\frac{1}{4} x e^{4x} \right]_0^1 - \frac{1}{4} \int_0^1 e^{4x} \, dx \\ &= \frac{1}{4} e^4 - \frac{1}{4} \left[\frac{1}{4} e^{4x} \right]_0^1 \\ &= \frac{1}{4} e^4 - \frac{1}{16} (e^4 - 1) \\ &= \frac{1}{16} (3e^4 + 1) \\ &= 10.300 \text{ (to 3 d.p.)}. \end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & \int_{-\pi/4}^{\pi/4} x \sin(2x) \, dx \\
&= \left[x \left(-\frac{1}{2} \cos(2x) \right) \right]_{-\pi/4}^{\pi/4} \\
&\quad - \int_{-\pi/4}^{\pi/4} 1 \times \left(-\frac{1}{2} \cos(2x) \right) \, dx \\
&= -\frac{1}{2} \left[x \cos(2x) \right]_{-\pi/4}^{\pi/4} \\
&\quad + \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos(2x) \, dx \\
&= -\frac{1}{2} \left(\frac{\pi}{4} \cos\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{4} \cos\left(-\frac{\pi}{2}\right) \right) \right) \\
&\quad + \frac{1}{2} \left[\frac{1}{2} \sin(2x) \right]_{-\pi/4}^{\pi/4} \\
&= -\frac{1}{2}(0 - 0) + \frac{1}{4} \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) \right) \\
&= \frac{1}{4}(1 - (-1)) \\
&= \frac{1}{2} \\
&= 0.500 \text{ (to 3 d.p.)}.
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad & \int_1^4 x^2 \ln x \, dx \\
&= \int_1^4 (\ln x) x^2 \, dx \\
&= \left[(\ln x) \times \frac{1}{3} x^3 \right]_1^4 - \int_1^4 \frac{1}{x} \left(\frac{1}{3} x^3 \right) \, dx \\
&= \frac{1}{3} \left[x^3 \ln x \right]_1^4 - \frac{1}{3} \int_1^4 x^2 \, dx \\
&= \frac{1}{3} \left(4^3 \ln 4 - 1^3 \ln 1 \right) - \frac{1}{3} \left[\frac{1}{3} x^3 \right]_1^4 \\
&= \frac{1}{3} (64 \ln 4 - 0) - \frac{1}{9} \left[x^3 \right]_1^4 \\
&= \frac{64}{3} \ln 4 - \frac{1}{9} (4^3 - 1) \\
&= \frac{64}{3} \ln 4 - 7 \\
&= 22.574 \text{ (to 3 d.p.)}.
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad & \int_1^2 x^2 \ln(4x) \, dx \\
&= \int_1^2 (\ln(4x)) x^2 \, dx \\
&= \left[\ln(4x) \times \frac{1}{3} x^3 \right]_1^2 - \int_1^2 \frac{1}{x} \times \frac{1}{3} x^3 \, dx \\
&= \frac{1}{3} \left[x^3 \ln(4x) \right]_1^2 - \frac{1}{3} \int_1^2 x^2 \, dx \\
&= \frac{1}{3} (8 \ln 8 - \ln 4) - \frac{1}{3} \left[\frac{1}{3} x^3 \right]_1^2 \\
&= \frac{1}{3} (24 \ln 2 - 2 \ln 2) - \frac{1}{9} \left[x^3 \right]_1^2 \\
&= \frac{1}{3} \times 22 \ln 2 - \frac{1}{9} (8 - 1) \\
&= \frac{1}{9} (66 \ln 2 - 7) \\
&= 4.305 \text{ (to 3 d.p.)}.
\end{aligned}$$

$$\begin{aligned}
\text{(e)} \quad & \int_0^1 x^2 e^{5x} \, dx \\
&= \left[x^2 \times \frac{1}{5} e^{5x} \right]_0^1 - \int_0^1 2x \times \frac{1}{5} e^{5x} \, dx \\
&= \frac{1}{5} \left[x^2 e^{5x} \right]_0^1 - \frac{2}{5} \int_0^1 x e^{5x} \, dx \\
&= \frac{1}{5} (e^5 - 0) \\
&\quad - \frac{2}{5} \left(\left[x \times \frac{1}{5} e^{5x} \right]_0^1 - \int_0^1 1 \times \frac{1}{5} e^{5x} \, dx \right) \\
&= \frac{1}{5} e^5 - \frac{2}{25} \left[x e^{5x} \right]_0^1 + \frac{2}{25} \int_0^1 e^{5x} \, dx \\
&= \frac{1}{5} e^5 - \frac{2}{25} (e^5 - 0) + \frac{2}{25} \left[\frac{1}{5} e^{5x} \right]_0^1 \\
&= \frac{1}{5} e^5 - \frac{2}{25} e^5 + \frac{2}{125} \left[e^{5x} \right]_0^1 \\
&= \frac{3}{25} e^5 + \frac{2}{125} (e^5 - 1) \\
&= \frac{1}{125} (15e^5 + 2e^5 - 2) \\
&= \frac{1}{125} (17e^5 - 2) \\
&= 20.168 \text{ (to 3 d.p.)}.
\end{aligned}$$

Solution to Exercise 22

We integrate by parts twice.

We can begin by swapping the order of the two factors in the integrand, to avoid introducing fractions in the early stages:

$$\int_0^{\pi/4} e^{x/2} \cos(2x) \, dx = \int_0^{\pi/4} (\cos(2x)) e^{x/2} \, dx.$$

Integrating by parts twice gives

$$\begin{aligned} & \int_0^{\pi/4} (\cos(2x)) e^{x/2} \, dx \\ &= \left[\cos(2x) \times 2e^{x/2} \right]_0^{\pi/4} \\ &\quad - \int_0^{\pi/4} (-2 \sin(2x)) \times 2e^{x/2} \, dx \\ &= 2 \left[e^{x/2} \cos(2x) \right]_0^{\pi/4} + 4 \int_0^{\pi/4} \sin(2x) e^{x/2} \, dx \\ &= 2 \left(e^{\pi/8} \cos\left(\frac{\pi}{2}\right) - e^0 \cos 0 \right) + 4 \int_0^{\pi/4} \sin(2x) e^{x/2} \, dx \\ &= 2(0 - 1) + 4 \left(\left[\sin(2x) \times 2e^{x/2} \right]_0^{\pi/4} \right. \\ &\quad \left. - \int_0^{\pi/4} 2 \cos(2x) \times 2e^{x/2} \, dx \right) \\ &= -2 + 8 \left[e^{x/2} \sin(2x) \right]_0^{\pi/4} - 16 \int_0^{\pi/4} e^{x/2} \cos(2x) \, dx \\ &= -2 + 8 \left(e^{\pi/8} \sin\left(\frac{\pi}{2}\right) - e^0 \sin 0 \right) \\ &\quad - 16 \int_0^{\pi/4} e^{x/2} \cos(2x) \, dx \\ &= -2 + 8(e^{\pi/8} - 0) - 16 \int_0^{\pi/4} e^{x/2} \cos(2x) \, dx \\ &= -2 + 8e^{\pi/8} - 16 \int_0^{\pi/4} e^{x/2} \cos(2x) \, dx. \end{aligned}$$

Hence

$$17 \int_0^{\pi/4} (\cos(2x)) e^{x/2} \, dx = -2 + 8e^{\pi/8},$$

which gives

$$\begin{aligned} \int_0^{\pi/4} e^{x/2} \cos(2x) \, dx &= \frac{2}{17} (4e^{\pi/8} - 1) \\ &= 0.5793 \text{ (to 4 d.p.)}. \end{aligned}$$

(An alternative way to evaluate the definite integral here is to integrate by parts twice to find the indefinite integral of $e^{x/2} \cos(2x)$, and then evaluate the definite integral directly.)

Solution to Exercise 23

(a) Use integration by substitution.

Let $u = 4x^3$; then $\frac{du}{dx} = 12x^2$. So

$$\begin{aligned} & \int x^2 \sin(4x^3) \, dx \\ &= \frac{1}{12} \int \sin(4x^3) \times 12x^2 \, dx \\ &= \frac{1}{12} \int \sin u \, du \\ &= -\frac{1}{12} \cos u + c \\ &= -\frac{1}{12} \cos(4x^3) + c. \end{aligned}$$

(b) Use integration by parts.

$$\begin{aligned} & \int x \sin(6x) \, dx \\ &= x \left(-\frac{1}{6} \cos(6x) \right) - \int 1 \times \left(-\frac{1}{6} \cos(6x) \right) \, dx \\ &= -\frac{1}{6} x \cos(6x) + \frac{1}{6} \int \cos(6x) \, dx \\ &= -\frac{1}{6} x \cos(6x) + \frac{1}{6} \times \frac{1}{6} \sin(6x) + c \\ &= \frac{1}{36} \sin(6x) - \frac{1}{6} x \cos(6x) + c. \end{aligned}$$

(c) Use integration by substitution.

Let $u = 8 - x^4$; then $\frac{du}{dx} = -4x^3$. So

$$\begin{aligned} & \int \frac{x^3}{(8 - x^4)^6} \, dx \\ &= -\frac{1}{4} \int \frac{1}{(8 - x^4)^6} (-4x^3) \, dx \\ &= -\frac{1}{4} \int \frac{1}{u^6} \, du \\ &= -\frac{1}{4} \int u^{-6} \, du \\ &= -\frac{1}{4} \times \frac{1}{-5} u^{-5} + c \\ &= \frac{1}{20u^5} + c \\ &= \frac{1}{20(8 - x^4)^5} + c. \end{aligned}$$

- (d) Expand the integrand.

$$\begin{aligned}
& \int \frac{5x^3 - 1}{2x} dx \\
&= \int \left(\frac{5x^3}{2x} - \frac{1}{2x} \right) dx \\
&= \int \left(\frac{5}{2}x^2 - \frac{1}{2} \times \frac{1}{x} \right) dx \\
&= \frac{5}{2} \times \frac{1}{3}x^3 - \frac{1}{2} \ln|x| + c \\
&= \frac{1}{6}(5x^3 - 3 \ln|x|) + c.
\end{aligned}$$

- (e) Use the rule for integrating a function of a linear expression.

$$\int \cos(4x - 5) dx = \frac{1}{4} \sin(4x - 5) + c.$$

Solution to Exercise 24

- (a) Use integration by parts.

$$\begin{aligned}
& \int_0^{\pi/6} x \cos(6x) dx \\
&= \left[x \times \frac{1}{6} \sin(6x) \right]_0^{\pi/6} - \int_0^{\pi/6} 1 \times \frac{1}{6} \sin(6x) dx \\
&= \frac{1}{6} \left[x \sin(6x) \right]_0^{\pi/6} - \frac{1}{6} \int_0^{\pi/6} \sin(6x) dx \\
&= \frac{1}{6} \left(\frac{\pi}{6} \sin \pi - 0 \right) - \frac{1}{6} \left[-\frac{1}{6} \cos(6x) \right]_0^{\pi/6} \\
&= 0 + \frac{1}{36} \left[\cos(6x) \right]_0^{\pi/6} \\
&= \frac{1}{36} (\cos \pi - \cos 0) \\
&= \frac{1}{36} (-1 - 1) \\
&= -\frac{1}{18} \\
&= -0.0556 \text{ (to 4 d.p.)}.
\end{aligned}$$

- (b) Use integration by substitution, expressing the extra factor
- $x + 2$
- in terms of
- u
- .

$$\text{Let } u = x + 1; \text{ then } \frac{du}{dx} = 1.$$

Rearranging the equation $u = x + 1$ gives $x = u - 1$, so

$$x + 2 = u - 1 + 2 = u + 1.$$

Also, $x = 1$ gives $u = 2$, and $x = 2$ gives $u = 3$.

So

$$\begin{aligned}
& \int_1^2 \frac{x + 2}{(x + 1)^2} dx \\
&= \int_2^3 \frac{u + 1}{u^2} du \\
&= \int_2^3 \left(\frac{1}{u} + \frac{1}{u^2} \right) du \\
&= \int_2^3 \left(\frac{1}{u} + u^{-2} \right) du \\
&= \left[\ln u + \frac{1}{-1} u^{-1} \right]_2^3 \\
&= \left[\ln u - \frac{1}{u} \right]_2^3 \\
&= \left(\ln 3 - \frac{1}{3} \right) - \left(\ln 2 - \frac{1}{2} \right) \\
&= \ln \frac{3}{2} + \frac{1}{6} \\
&= \frac{1}{6} + \ln \frac{3}{2} \\
&= 0.5721 \text{ (to 4 d.p.)}.
\end{aligned}$$